From Poisson to Lévy Process

Demystifying Lévy Measure and Compensation of Small Jumps

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Table of Contents

Introduction I

This material aims to provide basic insights into the jumps in Lévy processes, tailored for engineers and scientists from non-mathematical backgrounds.

You will learn:

1. Poisson and compound Poisson processes, including their superposition and thinning;
2. How the Lévy measure determines the intensity of jumps;
3. Why compensation of small jumps is necessary and how this compensation works.

- Measure theory is not used.
- We focus exclusively on jump processes and do not cover Brownian motion.

Definition 1.1

A Poisson process N_t (or $N(t)$) with intensity (rate parameter) $\lambda > 0$ is defined by the following formula, where $\mathbb{P}$ denotes probability

$$\mathbb{P}\left[N_t = k\right] = e^{-\lambda t}\frac{(\lambda t)^k}{k!}. \tag{1.1}$$

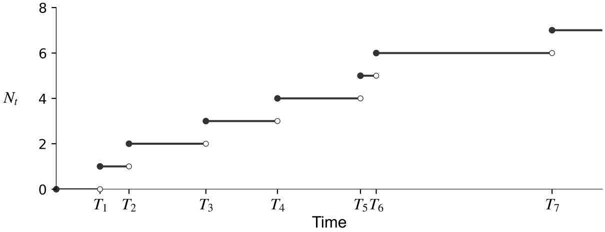

Figure 1.1 Example of Poisson process.

Proposition 1.2 (Superposition or sum of Poisson processes)

- Let $N_1(t)$ and $N_2(t)$ be two independent Poisson processes with rate parameters λ_1 and λ_2, respectively.
- Then the sum (or superposition) $N(t) = N_1(t) + N_2(t)$ is also a Poisson process with rate parameter $\boxed{\lambda = \lambda_1 + \lambda_2}$.

Proposition 1.3 (Thinning of Poisson process)

- Consider a Poisson process N_t with rate parameter λ.
- Suppose the occurrences consist of several categories, one of which (say, category A) occurs randomly with probability p_{A} among the occurrences.
- Then, the process restricted to category A is also a Poisson process with rate parameter $\boxed{p_{\mathrm{A}}\lambda}$.
- This construction of the new Poisson process is called thinning, and the resulting process is sometimes referred to as a thinned process.

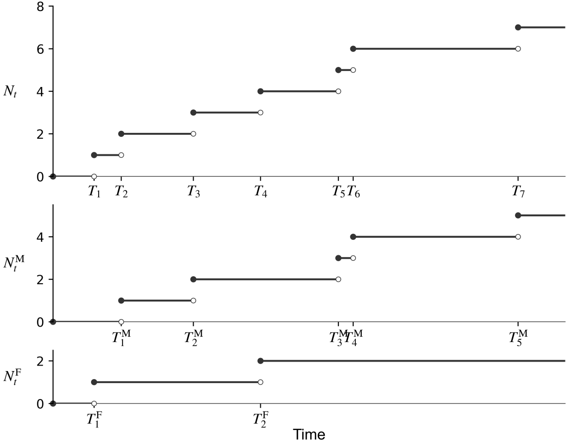

Figure 1.2 Example of thinning of Poisson process: N_t^{M} and N_t^{F} are created by thinning from N_t. Conversely, the superposition of N_t^{M} and N_t^{F} forms N_t. Let the rate parameters of N_t, N_t^{M}, and N_t^{F} be λ, λ^{M}, and λ^{F}, respectively, and the probabilities of realization of M and F be p and $1-p$, respectively. Then$\lambda = \lambda^{\mathrm{M}} + \lambda^{\mathrm{F}} = \lambda p + \lambda(1-p)$.

Definition 1.4 (Characteristic function)

The characteristic function of a random variable X is defined as follows:

$$\varphi_X(\xi) := \mathbb{E}\left[e^{i\xi X}\right]. \tag{1.2}$$

Remark 1.5

- The characteristic function determines the distribution, and vice versa.
- The notation $\varphi(X;\xi)$ is also used.
- If the random variable has a probability density function, then the characteristic function is the Fourier transform of the density function,

$$\mathbb{E}\left[e^{i\xi X}\right] = \int_{\mathbb{R}} e^{i\xi x}\, f(x)\, dx. \tag{1.3}$$

  In this case, the notation $\hat{f}(\xi) = \mathbb{E}\left[e^{i\xi X}\right]$ is also used, where $\hat{f}$ is the usual notation for the Fourier tranform of f.

Proposition 1.6 (Characteristic function of sum of random variables)

1. Let X and Y be independent random variables. Then the characteristic function of the random variable $X+Y$ is given by the following equation

$$\begin{aligned}\varphi_{X+Y}(\xi) &= \mathbb{E}\left[e^{i\xi(X+Y)}\right] = \mathbb{E}\left[e^{i\xi X}e^{i\xi Y}\right] \\ &\underset{\text{independence}}{=} \mathbb{E}\left[e^{i\xi X}\right]\mathbb{E}\left[e^{i\xi Y}\right] = \varphi_X(\xi)\varphi_Y(\xi).\end{aligned} \tag{1.4}$$

2. Let $Y_n = \sum_{i=1}^n X_i$, where X_i's are independent and identically distributed (iid). Then the characteristic function of Y_n is givewn by the following equation

$$\begin{aligned}\varphi\left(Y_n;\xi\right) &= \varphi\left(\sum_{i=1}^n X_i;\xi\right) = \mathbb{E}\left[e^{i\xi X_1}\cdots e^{i\xi X_n}\right] \\ &\underset{\text{independence}}{=} \mathbb{E}\left[e^{i\xi X_1}\right]\cdots\mathbb{E}\left[e^{i\xi X_n}\right] = \left\{\mathbb{E}\left[e^{i\xi X_i}\right]\right\}^n \\ &= \varphi(X_i;\xi)^n.\end{aligned} \tag{1.5}$$

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Definition 1.7

A compound Poisson process Y_t, with $\{X_i\}$ iid with density $f_X : \mathbb{R} \to \mathbb{R}_+$ is defined by the following equation:

$$Y_t = \sum_{i=1}^{N_t} X_i, \quad X_i \sim f_X, \quad \mathbb{P}(N_t = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}. \tag{1.6}$$

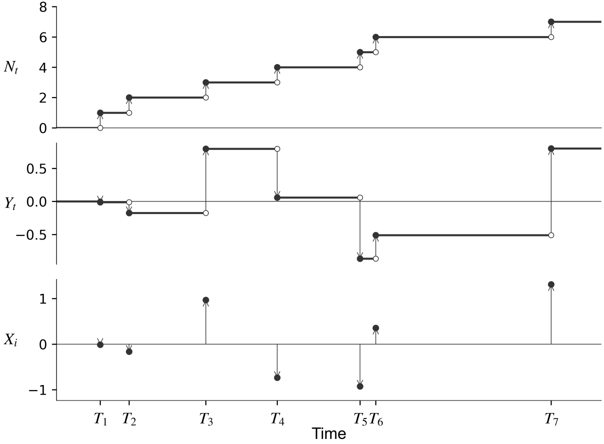

Figure 1.3 Comparion of Poisson and compound Poisson processes: N_t is a Poisson process, and $Y_t = \sum_{i=1}^{N_t} X_i$ is a compound Poisson process, where $X_i \sim f_X$. In the Poisson process, the jump size is always one. In contrast, in the compound Poisson process, the jump sizes are random variables.

Proposition 1.8 (Characteristic function of Poisson process)

$$\varphi_{N_t}(\xi) = \mathbb{E}\left[e^{i\xi N_t}\right] \exp\left\{t\lambda\left(e^{i\xi} - 1\right)\right\} \tag{1.7}$$

Theorem 1.9 (Characteristic function of compound Poisson process)

$$\varphi(Y_t;\xi) = \mathbb{E}\left[e^{i\xi Y_t}\right] = \exp\left[t\lambda\int_{\mathbb{R}}\left(e^{i\xi x} - 1\right) f(x)\,dx\right] \tag{1.8}$$

Proof of Proposition 1.8.

$$
\begin{aligned}
\varphi_{N_t}(\xi) = \varphi(N_t;\xi) = \mathbb{E}\left[e^{i\xi N_t}\right] &= \sum_{k=0}^{\infty} e^{i\xi k} e^{-\lambda t}\frac{(\lambda t)^k}{k!} \\
&= e^{-\lambda t}\sum_{k=0}^{\infty}\frac{\left(e^{i\xi}\lambda t\right)^k}{k!} = e^{-\lambda t}e^{\lambda t e^{i\xi}} = \exp\left\{\lambda t\left(e^{i\xi}-1\right)\right\}
\end{aligned}
\tag{1.9}
$$

□

Lemma 1.10

1. (Conditional expectation) Let X and Z be random variables, then

$$
\mathbb{E}\left[\mathbb{E}\left[X|Z\right]\right] = \mathbb{E}\left[X\right]. \tag{1.10}
$$

2.

$$
1 = \int_{\mathbb{R}} f_X(x)\,dx = \int_{\mathbb{R}} f_X(dx) \tag{1.11}
$$

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Proof of Theorem 1.9.

$$
\begin{aligned}
\mathbb{E}\left[e^{i\xi Y_t}\right] \underset{(1.10)}{=} \mathbb{E}\left[\mathbb{E}\left[e^{i\xi Y_t}\middle| N_t\right]\right] &= \mathbb{E}\left[\mathbb{E}\left[e^{i\xi \sum_{i=1}^{N_t} X_i}\right]\right] \\
&= \mathbb{E}\left[\left(\mathbb{E}\left[e^{i\xi X_1}\right]\right)^{N_t}\right] = \sum_{n=0}^{\infty} \frac{e^{-\lambda t}\left(\lambda t\right)^n \left(\hat{f}_X(\xi)\right)^n}{n!} \\
&= \exp\left\{\lambda t\left(\hat{f}_X(\xi) - 1\right)\right\} \\
&\underset{(1.11)}{=} \exp\left\{t\lambda \int_{\mathbb{R}} \left(e^{i\xi x} - 1\right) f_X(x)\, dx\right\}
\end{aligned}
\tag{1.12}
$$

□

Compound Poisson Process • Jumps in a Certain Range I

- Let us consider a compound Poisson process $Y_t^{[1]}$ with intensity $\lambda^{[1]}$ and jump size $X_i^{[1]}$ following the density $f_X^{[1]}(x)$.
- Suppose we are interested only in jumps of magnitude within the range $[\zeta, \zeta + \Delta\zeta)$.
- Then the jumps within the range $[\zeta, \zeta + \Delta\zeta)$ can be considered as a (thinned) compound Poisson process (denoted Y_t^{S1}) with intensity:

$$\lambda^{[1]}\mathbb{P}^{[1]}\left[\{X_i^{[1]} \in [\zeta, \zeta + \Delta\zeta)\}\right] = \lambda^{[1]} \int_{\zeta}^{\zeta+\Delta\zeta} f_X^{[1]}(x)\, dx. \qquad (1.13)$$

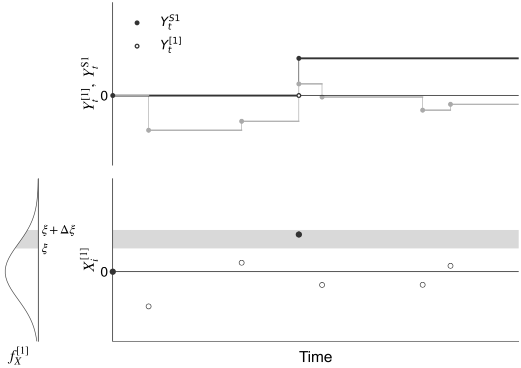

Figure 1.4 Compound Poisson process $Y_i^{[1]}$ with jumps $X_i^{[1]}$ and thinned process Y_t^{S1}, constructed by selecting only the jumps within the range $[\zeta, \zeta + \Delta\zeta)$.

Compound Poisson Process • Jumps in a Certain Range III

- Consider two compound Poisson processes $Y_t^{[1]}$ and $Y_t^{[2]}$ with intenisities $\lambda^{[1]}$ and $\lambda^{[2]}$, respectively, and jump sizes following densities $f_X^{[1]}$ and $f_X^{[2]}$, respectively.
- By selecting jumps of magnitude only within the range $[\zeta, \zeta + \Delta\zeta)$ from $Y_t^{[1]}$ and $Y_t^{[2]}$, we obtain two compound Poisson processes Y_t^{S1} and Y_t^{S2}.
- The superposition Y_t^{S12} of Y_t^{S1} and Y_t^{S2} is also a compound Poisson process, with intensity given by:

$$
\begin{aligned}
&\Lambda^{[1,2]}(\zeta, \Delta\zeta) \\
&\quad = \lambda^{[1]}\mathbb{P}^{[1]}\left[X_i^{[1]} \in [\zeta, \zeta + \Delta\zeta)\right] + \lambda^{[2]}\mathbb{P}^{[2]}\left[X_j^{[2]} \in [\zeta, \zeta + \Delta\zeta)\right] \\
&\quad = \lambda^{[1]}\int_{\zeta}^{\zeta+\Delta\zeta} f_X^{[1]}(x)\, dx + \lambda^{[2]}\int_{\zeta}^{\zeta+\Delta\zeta} f_X^{[2]}\, dx.
\end{aligned}
\tag{1.14}
$$

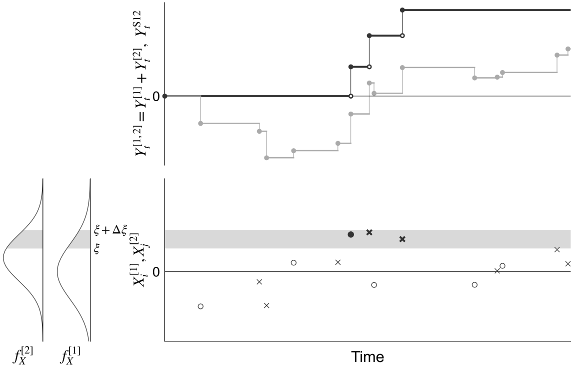

Figure 1.5 Superposition $Y_t^{[1,2]}$ of compound Poisson processes $Y_t^{[1]}$ and $Y_t^{[2]}$ and the superposition Y_t^{S12} of two thinned processes constructed by selecting jumps within the range $[\zeta, \zeta + \Delta\zeta)$.

Compound Poisson Process • Jumps in a Certain Range V

- It is known that we can superpose as many as countable processes, with the intensity given by:

$$\Lambda(\zeta, \Delta\zeta) = \sum_{i=1}^{\infty} \lambda^{[i]} \mathbb{P}^{[i]} \left[X_{j_i}^{[i]} \in [\zeta, \zeta + \Delta\zeta) \right]. \tag{1.15}$$

- To specify the intensity of the jumps in range $[\zeta, \zeta + \Delta\zeta)$ in the superposed process Y_t^{S}, we only need to determine $\Lambda(\zeta, \Delta\zeta)$, not necessarily all $\lambda^{[i]}$ and $\mathbb{P}^{[i]}$.
- For this purpose, we introduce a new function (measure), ν as follows:

$$\Lambda(\zeta, \Delta\zeta) = \int_{\zeta}^{\zeta+\Delta\zeta} \nu(x) dx. \tag{1.16}$$

Theorem 1.11

Consider an (at most countable) mixture Y_t of compound Poisson processes, where the intensity is a function of jump size: the jumps in the range $[x, x+dx)$ occur at intensity $\nu(x)dx$. Then the characteristic function of Y_t is given by the following equation:

$$\mathbb{E}\left[e^{i\xi Y_t}\right] = \exp\left\{t \int_{\mathbb{R}} \left(e^{i\xi x} - 1\right) \nu(x)\, dx\right\}. \tag{1.17}$$

Definition 1.12

We call ν in Eq. (1.17) a **Lévy measure**.

Remark 1.13

- For a constant λ, $\nu(x) = \lambda f_X(x)$, where f_X is the density of jump size.
- Under measure theory, $\nu(x)\, dx$ in Eq. (1.17) is usually written as $\nu(dx)$.
- We typically impose the condition $\nu(0) = 0$ to prevent jumps of size zero.

Remark 1.14

- We impose the following conditions on the Lévy measure

$$\nu(0) = 0 \quad \text{and} \tag{1.18}$$

$$\int_{\mathbb{R}} (\nu(x) \wedge 1)\, dx < \infty, \tag{1.19}$$

  where $a \wedge b$ means $\min(a, b)$.

- Eq. (1.18) is used to elimate jumps of size zero, wile Eq. (1.19) ensures the integral in Eq. (1.27) is well-defined.
- Some textbooks set the range of integration to $\mathbb{R} \setminus \{0\}$ instead of $\mathbb{R}$. In our settings, we assume $\nu(0) = 0$ and hence this restriction of the range is not necessary.

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- The author of this material believes that the compensation in Lévy processes is best understood through concrete examples.
- To illustrate this, we use the symmetric α-stable process, which is frequently observed in real-world applications.

Example 1.15

- Let $\nu(x) = C/|x|^{\alpha+1}$, $\alpha \in (0,2)$, which is the Lévy measure of the symmetric α-stable process.
- Taylor expansion of $e^{i\xi x}$ is

$$e^{i\xi x} = 1 + i\xi x + \frac{1}{2}(i\xi x)^2 + \frac{1}{6}(i\xi x)^3 \cdots .$$

- Thus

$$\begin{aligned}\frac{1}{C}\left(e^{i\xi x} - 1\right)\nu(x) &= \left(e^{i\xi x} - 1\right)\frac{1}{|x|^{1+\alpha}} \\ &= (i\xi)\frac{x}{|x|^{1+\alpha}} + \frac{(i\xi)^2}{2}\frac{x^2}{|x|^{1+\alpha}} + \frac{(i\xi)^3}{6}\frac{x^3}{|x|^{1+\alpha}} + \cdots .\end{aligned} \tag{1.20}$$

- As $0 < \alpha < 2$, the integral of the term $(i\xi)x/|x|^{1+\alpha}$ may diverge, while the integrals of the other terms converge.

Lévy Process • Compensated Process III

- In order to prevent the integral $\int_{\mathbb{R}} \left(e^{i\xi x} - 1\right) \nu(x) dx$ from diverging, we subtract the following term from the integrand:

$$(i\xi x)\nu(x) = C \frac{i\xi x}{|x|^{1+\alpha}}. \tag{1.21}$$

- As the first term of (1.20) explodes only at $x = 0$, the compensation is necessary only for very small values of $|x|$. Let $\eta > 0$ be small ($\eta \le 1$, and it is customary to choose $\eta = 1$), and let us define the characteristic function of the compensated process as follows (where $\mathbf{1}$ is the indicator function):

$$\begin{aligned} \varphi(Y_t) := \exp\Bigg\{ t \int_{|x| \le \eta} \left(e^{i\xi x} - 1 - i\xi x\right) \nu(x)\, dx \\ + t \int_{|x| > \eta} \left(e^{i\xi x} - 1\right) \nu(x)\, dx \Bigg\} \quad (1.22) \\ = \exp\left\{ t \int_{\mathbb{R}} \left(e^{i\xi x} - 1 - i\xi x \mathbf{1}_{|x| \le \eta}\right) \nu\, dx \right\}. \end{aligned}$$

- In (1.20), terms $x^k/|x|^{1+\alpha}$ are odd functions if k is odd, and even functions if k is even.
- Thus, in the integration of $\left(e^{i\xi x} - 1 - i\xi x\right)$, odd terms cancel and we have

$$\begin{aligned} &\frac{1}{C}\int_{|x|\le\eta}\left(e^{i\xi x} - 1 - i\xi x\right)\nu(x)\,dx \\ &= \int_{|x|\le\eta}\left\{\frac{(i\xi)^2}{2}\frac{x^2}{|x|^{1+\alpha}} + \frac{(i\xi)^4}{4!}\frac{x^4}{|x|^{1+\alpha}} + \cdots\right\}dx. \end{aligned} \tag{1.23}$$

- As can be seen, the integrand contains only real terms, and the resulting integral will be a real number. We have the following result:

$$\int_{|x|\le\eta}\left(e^{i\xi x} - 1 - i\xi x\right)\nu(x)\,dx = -C_2\xi^2 + C_4\xi^4 + \cdots, \tag{1.24}$$

where $C_2, C_4, \ldots$ are positive real numbers.

Lévy Process • Compensated Process V

- As $\eta \leq 1$, $|C_4| \ll |C_2|$, we have the follwoing approximation:

$$\int_{|x|\leq\eta} \left(e^{i\xi x} - 1 - i\xi x\right) \nu(x)\, dx \approx -C_2\xi^2. \tag{1.25}$$

- $\exp\left(-tC_2\xi^2\right)$ with $C_2 > 0$ is the characteristic function of a Brownian motion.

- So far, we have constructed a stochastic process that consists of jumps, where different magnitudes of jumps have different intensities of occurrence. The characteristic function of this process is expressed as follows:

$$\begin{aligned}\varphi_{Y_t}(\xi) &= \mathbb{E}\left[e^{i\xi Y_t}\right] \\ &= \exp\left\{\int_{\mathbb{R}}\left(e^{i\xi x} - 1 - i\xi x \mathbf{1}_{|x|\leq\eta}\right)\nu(x)dx\right\}.\end{aligned} \tag{1.26}$$

- A Lévy process is constructed by adding a Bronian process and a drift term to this jump process.

Theorem 1.16 (Lévy–Khintchine representation)

The characteristic function of the one-dimensional Lévy process Y_t is given by:

$$\varphi_{Y_t}(\xi) = \exp\left\{ai\xi - \frac{1}{2}\sigma^2\xi^2 + \int_{\mathbb{R}}\left(e^{i\xi x} - 1 - i\xi x \mathbf{1}_{|x|\leq\eta}\right)\nu(x)dx\right\}. \tag{1.27}$$

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Lévy Process • Conclusions I

- You have now learned the jump process part of the Lévy process, including:
  - Poisson and compound Poisson processes;
  - Superposition of compound Poisson processes;
  - Lévy measure;
  - Compensation for (potentially infinitely many) small jumps.
- Therefore, if you also learn Brownian motion, you will understand the Lévy process.
- We have skipped many mathematical subtleties, such as measure theory and right-continuity. However, the author of this material believes that what you have learned is sufficient for utilizing the Lévy process in real-life applications.

References

1. Sato, K (1999) Lévy process and infinitely divisible distributions, Cambridge University Press, Cambridge, UK.
2. Cont, R and Tankov, P (2003) Financial modelling with jump processes, Chapman and Hall/CRC, New York, USA.